

Week 10

Theorem: Let $X = \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Let γ optimal coupling for W_p .

Then, a Wasserstein geodesic between μ and ν is $\mu_t = \pi_t \# \gamma$,

where $\pi_t(x, y) = (1-t)x + ty$.

Def: F is geodesically convex if $\forall x, y, \exists \eta: [0, 1] \rightarrow X$ constant speed geodesic from x to y s.t. $F \circ \eta: [0, 1] \rightarrow \mathbb{R} \cup \{\infty\}$ is convex, i.e.,
$$F(\eta(t)) \leq (1-t)F(x) + tF(y).$$

→ lower semi-continuous: $\mu_n \rightarrow \mu$ in $(\mathcal{P}_p(\mathbb{R}^d), W_p) \Rightarrow U(\mu) \leq \liminf_n U(\mu_n)$

Convex and l.s.c. functionals on $\mathcal{P}_p(\mathbb{R}^d)$

Internal energy:
$$U(\mu) = \begin{cases} \int U(\rho) dx & \text{if } \mu = \rho(x) dx \\ +\infty & \text{otherwise.} \end{cases}$$

Potential energy: $V(\mu) = \int V d\mu$, $V: \mathbb{R}^d \rightarrow \mathbb{R}$ given.

Interaction energy: $W(\mu) = \int W(x, y) d\mu(x) d\mu(y)$, $W: \mathbb{R}^d \rightarrow \mathbb{R}$ given.

⚠ Convexity is NOT w.r.t the linear structure: $(U(t\mu + (1-t)\nu) \leq tU(\mu) + (1-t)U(\nu))$ ~~NO!~~

Displacement convexity

Def: Let X be a geodesic metric space, $F: X \rightarrow \mathbb{R} \cup \{\infty\}$, $\lambda \in \mathbb{R}$.

F is λ -geodesically convex (W_2 -convex or displacement convex; $\lambda = 0$)

if $\forall x, y \in X$, $\exists \eta: [0, 1] \rightarrow X$ constant speed geodesic between x and y such that: $F \circ \eta: [0, 1] \rightarrow \mathbb{R}$ is λ -convex, i.e.,

$$F(\eta(t)) \leq (1-t)F(x) + tF(y) - \frac{\lambda}{2} t(1-t) d^2(x, y) \quad \forall t \in [0, 1].$$

Remark (λ -convexity in $\mathbb{R}^d$): $F: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ λ -convex $\Leftrightarrow$

$$\Leftrightarrow F(x) - \frac{\lambda}{2} |x|^2 \text{ is convex} \Leftrightarrow D^2 F \geq \lambda \text{Id}$$

Indeed: F λ -convex $\Leftrightarrow F((1-t)x+ty) \leq (1-t)[F(x) - \frac{\lambda}{2}|x|^2] + t[F(y) - \frac{\lambda}{2}|y|^2] + \frac{t\lambda}{2}|y|^2 + (1-t)\frac{\lambda}{2}|x|^2 - \frac{\lambda}{2}t(1-t)|x-y|^2 + \frac{\lambda}{2}|(1-t)x+ty|^2 - \frac{\lambda}{2}|(1-t)x+ty|^2$ "0"

$\Leftrightarrow F - \frac{\lambda|x|^2}{2}$ convex $\Leftrightarrow D^2F - \lambda Id \geq 0$.

Convexity of potential energy

Theorem: $V: \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is λ -convex $\Leftrightarrow V: \mathcal{P}_2(X) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is λ -convex.

Proof: $\Rightarrow$ We claim that: $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, γ optimal

between μ_0 and μ_1 , μ_t geodesic $((1-t)\pi_1 + t\pi_2) \# \gamma$.

Then $t \mapsto \int V d\mu_t$ is convex,

$$\begin{aligned} \int V d\mu_t &= \int V((1-t)x+ty) d\gamma(x,y) \leq \int \left\{ (1-t)V(x) + tV(y) - \frac{\lambda}{2}t(1-t)|x-y|^2 \right\} d\gamma(x,y) \\ &= (1-t) \int V(x) d\mu_0(x) + t \int V(y) d\mu_1(y) - \frac{\lambda}{2}t(1-t)W_2^2(\mu_0, \mu_1). \end{aligned}$$

$\Leftarrow$ Apply convexity of V to δ_x and $\delta_y \Rightarrow$ convexity of V .

Convexity of internal energy $\int U(\rho) dx$ $\rho \in \mathcal{P}_2$ ^{important}

(HP1): $\begin{cases} U: [0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex, l.s.c., } U(0) = 0, \\ \limsup_{s \downarrow 0} \frac{U^-(s)}{s^\alpha} < +\infty \text{ for some } \alpha > \frac{d}{d+2} \end{cases}$ ^{negative part of U , $U^- = \max\{0, -U\}$} $d = \text{dimension}$

(HP2): $s \mapsto s^d U(s^{-d})$ convex and decreasing on $(0, \infty)$.

Remark: if (HP1) holds, then $\int U^-(\rho) dx < +\infty \quad \forall \rho \in L^1(\mathbb{R}^d)$. ^{we arbitrarily choose negative part finite, so that integral is well-defined.}

Indeed, by (HP1) $U^-(s) \leq C_1 s^\alpha + C_2 s$ ^{for some α , U convex}

$$\begin{aligned} \int U^-(\rho) &\leq C_2 \int \rho + C_1 \int (\rho(1+|x|^2))^\alpha \frac{1}{(1+|x|^2)^\alpha} \leq \\ &\leq \bar{C} + C_1 \left\{ \left[\int \rho(1+|x|^2) \right]^\alpha \left[\int \frac{dx}{(1+|x|^2)^{\frac{\alpha}{1-\alpha}}} \right]^{1-\alpha} \right\} < \infty \end{aligned}$$

integrable if $\frac{2\alpha}{1-\alpha} > d \Leftrightarrow \alpha > \frac{d}{d+2}$

absolutely continuous measures

Theorem: If (HP1)-(HP2) hold, then U is convex along geodesics in $\mathcal{P}_{a,2}(\mathbb{R}^d)$

Question: (HP1) & (HP2) hold for:

- ① $U(z) = z^\alpha$, α ?
- ② $U(z) = z \log z$?
- ③ $U(z) = -z^\alpha$ α ?

Proof: let $T: \mu_0 \rightarrow \mu_1$ optimal map, T^{-1} inverse, assume T and T^{-1} smooth.

let $\mu_t = T_t \# \mu_0$, $T_t = (1-t)I + tT$. Claim: $t \rightarrow U(\mu_t)$ is convex for $t \in [0,1]$.

Step 1: $\mu_t = \rho_t dx$ with $\rho_t = \frac{\rho_0}{JT_t} \circ T_t^{-1}$.

Indeed, $\forall \psi$:

$$\int \psi \rho_t dx = \int \psi d\mu_t = \int \psi \circ T_t \rho_0 = \int \psi \rho_0(T_t^{-1}(y)) \underbrace{|\det \nabla T_t^{-1}(y)|}_{\frac{1}{\det \nabla T_t} \circ T_t^{-1}} dy$$

Step 2: $t \mapsto U(\mu_t) = \int U\left(\frac{\rho_0}{JT_t} \circ T_t^{-1}\right) dx \stackrel{T_t^{-1}(x)=y}{=} \int U\left(\frac{\rho_0}{JT_t}\right) JT_t dx$ is convex.

At x fixed, define: $t \xrightarrow{F} \underbrace{JT_t(x)}_{\geq 0} \xrightarrow{G} JT_t U\left(\frac{\rho_0}{JT_t}\right)$

G is $G(s) = s^d U\left(\frac{\rho_0(x)}{s^d}\right)$ which is convex and decreasing.

$$F \text{ concave} \Leftrightarrow F\left(\frac{t_1+t_2}{2}\right) \geq \frac{F(t_1)+F(t_2)}{2} \Leftrightarrow$$

$$\Leftrightarrow \det^{1/d} \left(\underbrace{\left(1 - \frac{t_1+t_2}{2}\right)I}_{\geq 0} + \frac{t_1+t_2}{2} \underbrace{\nabla T}_{\geq 0} \right) \geq \det^{1/d} \left((1-t_1)I + t_1 \nabla T \right) + \det^{1/d} \left((1-t_2)I + t_2 \nabla T \right)$$

$\Leftarrow \det^{1/d}$ is a concave function (Step 3).

At fix x , $G \circ F$ is convex:

$$\underbrace{(G \circ F)((1-s)t_1 + st_2)}_{\geq (1-s)F(t_1) + sF(t_2)} \leq G((1-s)F(t_1) + sF(t_2)) \leq (1-s)G \circ F(t_1) + sG \circ F(t_2).$$

$\downarrow$ F concave $\downarrow$ G decreasing $\downarrow$ G convex

Step 3: $A \rightarrow (\det A)^{1/d}$ is concave, $A \in \text{Sym}_{\geq 0}^{d \times d}$ matrix.

We will show: $\det((1-t)A_0 + tA_1)^{1/d} \geq (1-t)\det^{1/d} A_0 + t\det^{1/d} A_1$ (*).

We first reduce to having $A_0 = I$ { divide by $\det^{1/d} A_0$ both sides:
 $\det((1-t)I + tA_0^{-1}A_1)^{1/d} \geq (1-t) + t\det^{1/d} A_0^{-1}A_1$
 Not a good idea: $A_0^{-1}A_1$ is not symmetric! }

A_0 symmetric $\rightarrow A_0 = MDM^{-1}$, D diagonal, M orthonormal:

$$\sqrt{A_0} := MD^{1/2}M^{-1}, \quad D^{1/2} = \begin{pmatrix} d_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & d_d^{1/2} \end{pmatrix} \quad \sqrt{A_0}\sqrt{A_0} = A_0$$

$$(*) \Leftrightarrow \det^{1/d} \left(\underbrace{A_0^{-1/2}}_X [(1-t)I + tA_0^{-1/2}A_1A_0^{-1/2}] \underbrace{A_0^{1/2}}_X \right) \geq (1-t)(\det^{1/d} A_0) \oplus + t\det^{1/d} \left(\underbrace{A_0^{-1/2}}_X A_0^{-1/2} A_1 \underbrace{A_0^{-1/2}}_X \underbrace{A_0^{1/2}}_X \right)$$

and $A_0^{-1/2}A_1A_0^{-1/2}$ is symmetric!

Diagonalize! Assume in $\oplus$ that $\underbrace{A_0^{-1/2}A_1A_0^{-1/2}}_{\text{diagonal}} (\lambda_1 \dots \lambda_d)$

$$\oplus \Leftrightarrow \prod_{i=1}^d ((1-t) + t\lambda_i)^{1/d} \geq (1-t) + t \prod_{i=1}^d \lambda_i^{1/d} \Leftrightarrow$$

$$\Leftrightarrow 1 \geq (1-t) \cdot \underbrace{\left[\prod_{i=1}^d \frac{1}{1-t+t\lambda_i} \right]^{1/d}}_{\text{AM-GM} \leq \text{AM}} + t \underbrace{\left[\prod_{i=1}^d \frac{\lambda_i}{1-t+t\lambda_i} \right]^{1/d}}$$

$$\frac{1}{d} \sum_{i=1}^d \frac{1}{1-t+t\lambda_i} + \frac{1}{d} \sum_{i=1}^d \frac{\lambda_i}{1-t+t\lambda_i} = 1 \quad \square$$

Application: Brunn-Minkowski inequality still compact

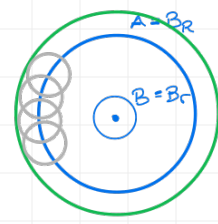
Theorem: $A, B \subseteq \mathbb{R}^d$ compact. $\mathcal{L}^d(A+B)^{1/d} \geq \mathcal{L}^d(A)^{1/d} + \mathcal{L}^d(B)^{1/d}$ (*)

Proof: Given $C \subseteq \mathbb{R}^d$, we associate $\mu_C := \frac{1}{\mathcal{L}^d(C)} \mathbb{1}_C dx$.

Take $\mathcal{U}(\mu) = -\int \rho^{1-\frac{1}{d}} dx$ / (3) in previous remark.

$$-\mathcal{U}(\mu_C) = \int_C \frac{dx}{\mathcal{L}^d(C)^{1-1/d}} = \mathcal{L}^d(C)^{1/d}$$

RHS of (*)



$A+B = B_{R+r}$

equality case for Brunn-Minkowski!

Let μ_t geodesic between μ_A and μ_B : $\frac{1}{2}[-\mathcal{U}(\mu_A) - \mathcal{U}(\mu_B)] \leq -\mathcal{U}(\mu_{1/2})$

We are now left to prove $-\mathcal{U}(\mu_{1/2}) \leq \mathcal{L}^d\left(\frac{A+B}{2}\right)^{1/d}$. Scale of space

Claim: ρ probability supported on $C \Rightarrow -\mathcal{U}(\rho) := \int \rho^{1-1/d} dx \leq \mathcal{L}^d(C)^{1/d}$.

With this claim we finish: $\mu_{1/2} = \frac{\underbrace{\chi_A + \chi_B}_{\text{on } A+B}}{2} \neq \mu_A$ is concentrated on $\frac{A+B}{2} = C$.

Proof of claim: $\int_C \rho^{1-1/d} dx = \mathcal{L}^d(C) \int_C \rho^{1-1/d} \frac{dx}{\mathcal{L}^d(C)} \leq \mathcal{L}^d(C) \left[\int_C \rho \frac{dx}{\mathcal{L}^d(C)} \right]^{1-1/d} = \mathcal{L}^d(C)^{1/d}$. concave $\varphi(\rho)$ ← Jensen